APPLICATION OF ASYMPTOTIC METHODS OF THE THEORY OF NONLINEAR OSCILLATIONS TO THE PROBLEM OF WAVE PROPAGATION

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The problem of the propagation of waves in an inhomogeneous medium is solved on the basis of the equation for a partial wave of the total field. After changing the independent variable x (the geometrical coordinate) to A(x) (the amplitude factor of a direct partial wave of the total field in the inhomogeneous medium) a modification of one of the asymptotic methods of the theory of nonlinear oscillations is applied.

It is well known that the differential equation

$$d^2\psi(k_0, x)/dx^2 + k^2(k_0, x)\psi(k_0, x) = 0 \tag{0.1}$$

plays an important part in the theory of propagation of acoustic [1] and electromagnetic [2] waves, that it describes a quasi-stationary temperature field relative to moving coordinates [3], and, on passing from $k(k_0, x)$ to the function $[\lambda^2 - V(k_0, x)]$ it describes the passage of matter waves through a potential barrier [4]. However, no method has so for been developed which allows one to find the connection between the function of the medium parameters $k = k(k_0, x)$ and the total field function $\psi = \psi(k_0, x)$, and which is general enough to allow one to determine $\psi(k_0, x)$ with any degree of accuracy when the parameter $k(k_0, x)$ is an arbitrary function of the coordinatex. The asymptotic method for the solution of wave propagation problems set out below allows one to determine, on the basis of a selected auxiliary function, the corresponding functions $\psi(k_0, x)$ and $k(k_0, x)$ simultaneously with any degree of accuracy specified in advance.

At the same time as solving the problem of wave propagation, the asymptotic method allows one to consider also the case of non-propagation of waves, when a solution occurs which is not oscillatory, but behaves like real exponents and describes bound states belonging to the discrete spectrum. This also lends confirmation to the assumption made in the appendix of [5] that there may be a smooth transition between the properties of the discrete spectrum and the properties of the quasi-stationary states.

The asymptotic method is based on equations obtained by the method of internal conditions [6,7], which may be regarded as a generalization of the well-known Oseen theorem in optics [8] to wave processes of a different nature. These equations for the direct $\alpha(x)$ and inverse $\beta(x)$ partial waves of the field $\psi(x)$ in an inhomogeneous medium have [9] the form

$$\alpha''(k_0, x) + p(k_0, x)\alpha'(k_0, x) + k(k_0, x)[k(k_0, x) - ip(k_0, x)]\alpha(k_0, x) = 0, \tag{0.2}$$

$$\beta''(k_0, x) + p(k_0, x)\beta'(k_0, x) + k(k_0, x)[k(k_0, x) + ip(k_0, x)]\beta(k_0, x) = 0, \tag{0.3}$$

where

$$p(k_0, x) = 2[\ln k(k_0, x)]' - [\ln k'(k_0, x)]', \tag{0.4}$$

and differentiation with respect to the coordinate x is indicated by a prime. The problem is solved in three stages.

In the first stage the amplitude A and phase φ factors are separated from the direct wave and a change is made from the variable x to the variable A(x). The second stage reduces to looking for the general solution of the differential equation (0.1) with the help of phase trajectories, and the third consists in passing back from A(x) to the independent variable x.

Besides the independent equations (0.2) and (0.3), the solution process also makes use of a system of coupled first-order equations

$$\alpha'(k_0, x) + [\varkappa(k_0, x) - ik(k_0, x)]\alpha(k_0, x) = \varkappa(k_0, x)\beta(k_0, x), \tag{0.5}$$

$$\beta'(k_0, x) + [\varkappa(k_0, x) + ik(k_0, x)]\beta(k_0, x) = \varkappa(k_0, x)\alpha(k_0, x),$$

$$2\varkappa(k_0, x) = [\ln k(k_0, x)]' \tag{0.6}$$

somewhat more general [6] than the initial equation (0.1).

1. The basic relations of the asymptotic method.

The general solution of Eq. (0.1) is expressed in terms of the nontrivial particular solution $[\alpha(x) + \beta(x)]$ in the following manner:

$$\psi(x) = [\alpha(x) + \beta(x)]\Big[C_1 + C_2\int_0^x [\alpha(x_1) + \beta(x_1)]^{-2}dx_1\Big] \tag{1.1}$$

where C_1 and C_2 are constants of integration determined from the initial conditions. We shall look for this solution in the form

$$\psi(x) = Y_1e^{i\Psi_1}[C_1 + C_2Y_2e^{-i\Psi_2}], \tag{1.2}$$

$$Y_1e^{i\Psi_1} = [\alpha(x) + \beta(x)], \qquad Y_2e^{-i\Psi_2} = \int_0^x [\alpha(x_1) + \beta(x_1)]^{-2}dx_1. \tag{1.3}$$

The particular solution $[\alpha(x) + \beta(x)]$ may be expressed in terms of $\alpha(x)$ only, taking (0.5), (0.6) into account,

$$[\alpha(x) + \beta(x)] = 2\left\{\frac{k(x)}{k'(x)}\alpha'(x) + \left[1 - i\frac{k^2(x)}{k'(x)}\right]\alpha(x)\right\}. \tag{1.4}$$

Keeping in mind the necessity of passing to the independent variable A(x) later on, we represent the wave $\alpha(x)$ in the form

$$\alpha(x) = A(x)e^{i\varphi(x)}, \tag{1.5}$$

and rewrite the particular solution (1.4) of Eq. (0.1) as

$$[\alpha(x) + \beta(x)] = 2\left|\frac{k(x)}{k'(x)}A(x)\right|\{L^2(x) + [\varphi'(x) - k(x)]^2\}^{1/2} \times \exp\left\{i\left[\varphi(x) + \operatorname{arc\,tg}\frac{\varphi'(x) - k(x)}{L(x)}\right]\right\}, \tag{1.6}$$

where

$$L(x) = A'(x)A^{-1}(x) + k'(x)k^{-1}(x). \tag{1.7}$$

Since the right side of (1.6) contains only the amplitude and phase factors of the wave $\alpha(x)$, we shall restrict ourselves in what follows to a consideration of Eq. (0.2). Using the method developed in the theory of nonlinear oscillations, which reduces [10,11] to a graphical construction of integral curves in the phase plane, we introduce into (0.2) expressions for $\alpha'(x)$ and $\alpha''(x)$ obtained by differentiating (1.5).

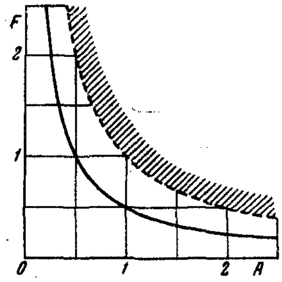

Fig. 1

The system of two differential equations connecting the amplitude A(x) and phase $\varphi(x)$ factors of the direct partial wave $\alpha(x)$ then has the form

$$A''(x) + p(x)A'(x) + \{k^2(x) - [\varphi'(x)]^2\}A(x) = 0, \quad (1.8)$$

$$\varphi''(x) + \left[2\frac{A'(x)}{A(x)} + p(x)\right]\varphi'(x) - k(x)\,p(x) = 0. \quad (1.9)$$

After making the change of variable $x \to A(x)$ we use this system to solve the wave propagation problem in a lossless inhomogeneous medium by the asymptotic method.

2. Description of the method of solving the propagation problem. We shall carry out a change of variable in the system (1.8), (1.9), taking the wave number to be a function of the amplitude factor A(x), i.e., k(x) = k[A(x)] = k(A),

$$\frac{dk}{dx} = \frac{dk}{dA}\frac{dA}{dx} = k^{\cdot}A', \qquad \frac{d^2k}{dx^2} = k^{\cdot}A'' + k^{\cdot\cdot}(A')^2.$$

We obtain

$$p(x) = p(A)\,A' - A''(A')^{-1}, \qquad p(A) = 2\frac{k^{\cdot}}{k} - \frac{k^{\cdot\cdot}}{k^{\cdot}} \quad (2.1)$$

for the coefficient p(x) determined by (0.4).

Introducing (2.1) into (1.8) and (1.9), we write

$$p(A)A^{-1}(A')^2 + k^2 = (\varphi')^2 \quad \text{or} \quad \varphi' = [p(A)A^{-1}(A')^2 + k^2]^{1/2}, \quad (2.2)$$

$$AA'\varphi'' - A(\varphi' - k)A'' + \{[2 + Ap(A)]\varphi' - Akp(A)\}(A')^2 = 0. \quad (2.3)$$

Differentiating φ' with respect to the coordinate x, we obtain

$$\varphi'' = A'\left[\frac{p}{A}A'' + \frac{1}{2A}\left(p^{\cdot} - \frac{p}{A}\right)(A')^2 + kk^{\cdot}\right]\left[\frac{p}{A}(A')^2 + k^2\right]^{-1/2}. \quad (2.4)$$

We set expressions φ' and φ'' in (2.3) and set A' = F, then

$$F^{-2}F^{\cdot} = p + \left[\left(k^{\cdot} + \frac{2k}{A}\right) + \frac{p}{Ak}\left(\frac{p^{\cdot}}{2p} + \frac{3}{2A} + p\right)F^2\right]\left[k - \left(\frac{p}{A}F^2 + k^2\right)^{1/2}\right]^{-1}.$$

Let θ = arc tg $F^{\cdot}$ be the angle of incidence of the tangent to the curve F = F(A) in the phase plane AOF. We then obtain the key equation of the asymptotic method of solving wave propagation problems, remembering that $F' = F^{\cdot}F$, in the form

$$\mathrm{tg}\,\theta = \{p(A) + [f_1(A) + f_2(A)F^2]\Phi^{-1}(A, F)\}F, \quad (2.5)$$

$$f_1(A) = k^{\cdot}(A) + 2k(A)A^{-1}, \quad (2.6)$$

$$f_2(A) = p(A)\,[Ak(A)]^{-1}\{p(A) + 3(2A)^{-1} + p^{\cdot}(A)\,[2p(A)]^{-1}\}, \quad (2.7)$$

$$\Phi(A, F) = k(A) - [p(A)A^{-1}F^2 + k^2(A)]^{1/2}. \quad (2.8)$$

It is not a difficult matter to find the functions $f_1(A)$, $f_2(A)$, Φ (A, F) and p(A) entering into Eq. (2.5), for the function k(A) = k[A(x)] corresponding to the function k = k(x) specified by the conditions of the given problem. Consequently, the angle θ may also be determined for an arbitrarily chosen point P(A, F) on the phase plane AOF, and subsequently a system of curves F = F(A) may be constructed which in accordance with the asymptotic method of the theory of nonlinear oscillations, will be called the direction field in the phase plane.

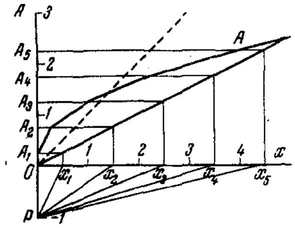

Fig. 2

We note three basic properties of the first stage in solving the problem which must be kept in mind in carrying out the first stage, i.e., in passing back from the variable A to the independent variable x.

(a) The system of curves F = F(A) of the direction field may easily be transformed to the system of curves $x^{\cdot} = f(A)$ and after integration (with respect to the variable A) transferred to the plane AOx. The system of curves obtained A = A(x) characterizes the dependence of the amplitude factor of the direct partial wave A(x) on the coordinate x; the curves of this system are shifted relative to each other along the Ox axis. This procedure enables us to ascertain immediately the amplitude factor A(x) of the direct partial wave $\alpha(x)$ entering into the particular solution of the initial equation (0.1) as a function of the initial coordinate x.

(b) While determining A(x), it is also possible to establish, from the chosen function k = k(A), the initial law of variation of wave number along the coordinate, i.e., to find the function k = k(x) corresponding to any of the functions A = A(x) found as described in (a). At the same time, an unambiguous connection is established between the law of variation of the parameter k = k(x) and the amplitude factor of the direct partial wave A(x) entering into the particular solution (1.6) of the initial differential equation (0.1).

(c) In order to find the phase factor of the direct partial wave $\varphi = \varphi(x)$ we must employ Eq. (2.2), which determines the gradient of the required phase factor $\varphi'(x)$. Taking into account that

$$\varphi' = \varphi^{\cdot} F$$

it is convenient to represent Eq. (2.2) in the form

$$\varphi^{\cdot} = \{p(A)A^{-1} + [k(A)F^{-1}(A)]^2\}^{1/2}. \qquad (2.9)$$

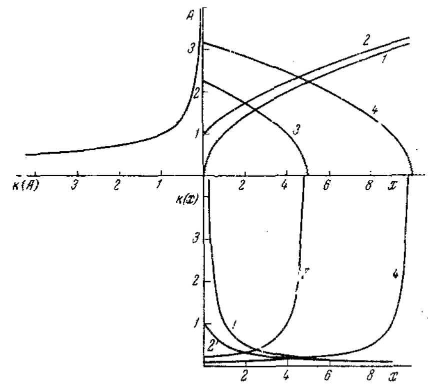

Fig. 3

After integration with respect to the variable A it is not difficult to obtain the function $\varphi = \varphi(A)$ for any curve F = F(A) of the previously constructed direction field. In accordance with (b) it is here possible, by choosing any specific curve A = A(x) and thus specifying the function k = k(x) unambiguously, to transform the function $\varphi = \varphi(A)$ obtained into the required function $\varphi = \varphi(x)$. Thus from the function k = k(A) selected it is not difficult to find both the law of variation of the properties of the inhomogeneous medium along the coordinate k = k(x) and the direct partial wave $\alpha(x) = A(x)e^{i\varphi(x)}$ of the total field in the medium. Of great importance is the possibility we now have of applying this to both direct and inverse variants of the wave propagation problem, which reduces to a choice of the function k = k(A) arising from the required law k = k(x) for the medium, or from the required total field $\psi(x)$ in the medium.

The second stage in solving the problem reduces to finding the general solution $\psi(x)$ of the initial equation (0.1). For this reason we change the variable in (1.6) from x to A just as when we were establishing with the help of the function k = k(A) the connection between the direct partial wave $\alpha(x)$ and the law of variation of the properties of the medium along the coordinate k = k(x), i.e., we pass from k(x), k' and φ' to K(A), $k^{\cdot}$ and $\varphi^{\cdot}$

$$[\alpha + \beta] = 2\left|\frac{k}{k^{\cdot}}A\right|(M^2 + N^2)^{1/2}\exp\left[i\left(\varphi + \operatorname{arc\,tg}\frac{M}{N}\right)\right] \qquad (2.10)$$

where the following symbols are employed for the auxiliary functions:

$$M = \left[\frac{p}{A} + \left(\frac{k}{F}\right)^2\right]^{1/2} - \frac{k}{F}, \qquad N = \frac{1}{A} + \frac{k^{\cdot}}{k}. \qquad (2.11)$$

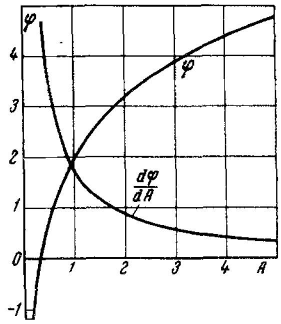

Fig. 4

Comparing (2.10) and (1.3), it is not difficult to establish that the functions Y_1, Ψ_1, Y_2, Ψ_2 appearing in the general solution (1.2) are equal to

$$Y_1 = 2\left|\frac{k}{k^{\cdot}}A\right|(M^2 + N^2)^{1/2}, \qquad \Psi_1 = \varphi + \operatorname{arc\,tg}\frac{M}{N}, \qquad (2.12)$$

$$Y_2 = (J_1^2 + J_2^2)^{1/2}, \qquad \Psi_2 = \operatorname{arc\,tg}\frac{J_2}{J_1}. \qquad (2.13)$$

Here

$$J_1 = \int_0^{A_1} R_1(A)\,dA, \qquad J_2 = \int_0^{A_1} R_2(A)\,dA,$$

$$R_1(A) = \cos 2\Psi_1\,[Y_1^2 F]^{-1},$$

$$R_2(A) = \sin 2\Psi_1\,[Y_1^2 F]^{-1}. \qquad (2.14)$$

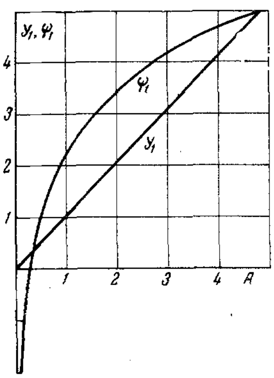

Fig. 5

Thus, in order to find the general solution $\psi(x)$ of the wave equation (0.1) starting from the function k = k(A) and any definite chosen curve F = F(A) from the previously constructed direction plane, it is necessary to transfer the graphs of $\varphi = \varphi(A)$ to the phase plane AOF (in accordance with paragraph (c) of the first stage of the solution), and also the graphs M = M(A) and N = N(A) from formulas (2.11). Further, the functions $Y_1 = Y_1(A)$ and $\Psi_1 = \Psi_1(A)$ must be found with the help of formulas (2.12). In order to find the functions $Y_2 = Y_2(A)$ and $\Psi_2 = \Psi_2(A)$, which are determined by the relations (2.13), we must first of all construct the curves $R_1 = R_1(A)$ and $R_2 = R_2(A)$ from (2.14).

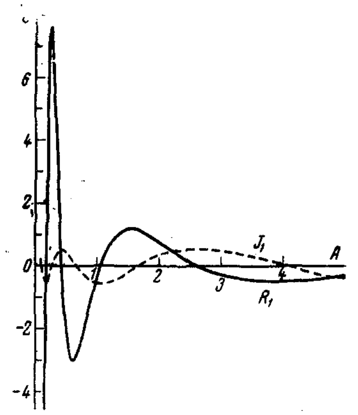

Fig. 6

In accordance with (1.2) the functions which are found Y_1, Ψ_1, Y_2 and Ψ_2 give a full description of the general solution $\psi(A)$ of the initial equation (0.1), which is represented as a function of the intermediate variable A(x).

In the third stage of the solution of the problem we pass back from the variable A(x) to the original independent variable x. It is here necessary to keep in mind that the law of variation of the wave vector along the coordinate k = k(x), corresponding to the general solution $\psi(A)$ found and to the chosen function k = k(A), was determined previously (see paragraph (b) of the first stage of the solution). After the change of variable from A to x, the general solution $\psi(x)$ of Eq. (0.1) is represented, according to (1.2), in the form of four functions $Y_1(x)$, $\Psi_1(x)$, $Y_2(x)$ and $\Psi_2(x)$, and corresponds to the appropriate function k = k(x).

3. Certain properties of the asymptotic method which allow the solution of the problem to be simplified. When using the asymptotic method for solving the problem of wave propagation in an inhomogeneous medium we should keep in mind that it is an especially simple matter to construct the so-called boundary trajectories on which the phase $\varphi(x)$ of the wave $\alpha(x)$ is constant. Actually the auxiliary function (2.8) is purely real in accordance with the requirements of the problem. Thus for the expression which appears in it under the root sign

$$(pA^{-1}F^2 + k^2) > 0.$$

By definition F = A', and so in accordance with (2.4) we obtain

$$(pA^{-1}F^2 + k^2) = (\varphi')^2 > 0. \qquad (3.1)$$

Consequently, the phase of the wave $\alpha(x)$ does not have an imaginary part (i.e., there is no damping when the wave propagates in the medium). It follows from (3.1) that in particular

$$F < k\,(-pA^{-1})^{-1/2}, \quad p\,(A) < 0.$$

The inverse condition

$$(pA^{-1}F^2 + k^2) < 0$$

assumes the case of purely imaginary phase mentioned in the introduction, when the wave does not propagate in the medium but is exponentially damped, i.e., when there is a nonoscillatory solution which describes bound states belonging to the discrete spectrum. This case is not considered here because of lack of space.

The intermediate case (critical condition) occurs when

$$(pA^{-1}F^2 + k^2) = 0.$$

This case corresponds to the condition $\varphi' = 0$, while the curves in the plane AOF described by the equation

$$F = k\,(-pA^{-1})^{-1/2} \qquad (3.2)$$

are the boundary trajectories, the construction of which is a considerable aid not only in establishing the connection with the function k = k(x), but also in finding the curves F = F(A). The basic properties of the curves F = F(A) forming the direction field are as follows:

a) The direction field in the plane AOF is symmetrical with respect to the OA axis, and so it suffices to consider only the region in which $F \geq 0$. Passing to the case $F < 0$ is completely equivalent to the specular reflection of the curve A = A(x) in the plane AOx relative to the OA axis.

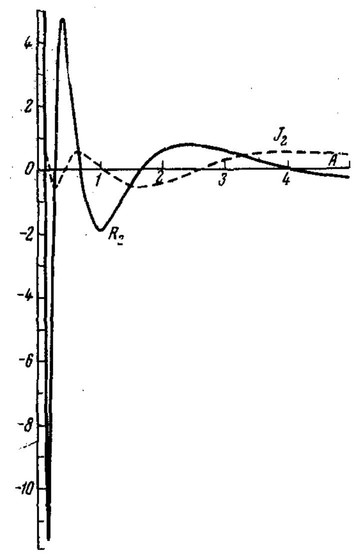

Fig. 7

b) Points at which $F' = 0$, i.e., points at which the tangents to the curve F = F(A) of the direction field are horizontal, are determined for $F \neq 0$, $F \neq \infty$ by the equation

$$F^2 = \frac{1}{f_2}\left\{-(kp + f_1) + \frac{p^3}{2Af_2} + p\left[\left(k - \frac{p^2}{2Af_2}\right)^2 - \frac{f_1 p}{f_2 A}\right]^{1/2}\right\}. \qquad (3.3)$$

c) On the other hand for the case $F \to 0$ corresponding to the OA axis, it follows from Eq. (2.5) that for $f_1 \neq 0$, $p \neq \infty$

$$\lim F' \to \infty \quad \text{for} \quad F \to 0,$$

and consequently that the tangents to the curves $F = F(A)$ are vertical at the points where they intersect the OA axis, while the points where $f_1 = 0$, $p = \infty$ require additional investigation.

Moreover,
$\lim F' \to \infty$ for $F \to \infty$, and $\lim F' \to \infty$ for $A \to 0$, i.e., the curves $F = F(A)$ have vertical asymptotes for $F \to \infty$ as well as for $A \to 0$.

d) It follows from the linearity of the wave propagation problem in an inhomogeneous medium without dispersion that when the parameter k(A) or its argument is multiplied by a constant quantity c it is only necessary to carry out the corresponding change of scale along the coordinate x (or of the amplitude of the direct partial wave). Actually, it follows from the main equation of the problem (2.5) that in passing from the parameter k(A) to the new parameter $k^\circ(A) = ck(A)$, where c is a constant, it suffices to change the function F(A) in the main equation to the function $F^\circ(A) = cF(A)$. The result of this is that when the parameter k(A) is changed to the parameter $k^\circ(A)$ the variable $A = A(x)$ must be changed to the variable $A^\circ = A(x/c)$, and k(x) to $k^\circ = ck(x/c)$. Similarly, the change of variable $k(A) \to k(cA)$ requires the change $A = A(x) \to A^\circ = c^{-1}A(x)$ while retaining the parameter $k = k(x)$.

4. An example of the solution of the wave propagation problem. We shall first of all consider the grapho-analytic variant of the asymptotic method as being more intuitive. We shall choose a function $k = k(A)$, which shows how the parameter k of the inhomogeneous medium depends on the coordinate x, in the following form:

$$k(A) = A^{-2}$$

i.e., we take that

$$k(x) = (ax + b)^{-1}. \tag{4.1}$$

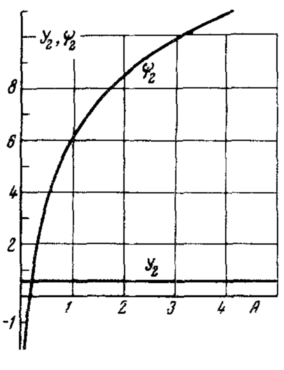

Fig. 8

Since $p(A) = -A^{-1}$, $f_1(A) = 0$, $f_2(A) = 0$, according to (2.1), (2.6), (2.7), the main equation (2.5), determining the angle of inclination θ of the tangents to the family of curves $F = F(A)$, is written in the form

$$\operatorname{tg}\theta = F'(A) = -FA^{-1}. \tag{4.2}$$

Thus the equation of the boundary trajectories (3.2) has the form

$$F(A) = k(A)[-p(A)A^{-1}]^{-1/2} = A^{-1}. \tag{4.3}$$

The boundary trajectory constructed from (4.3) is drawn as a broken line in Fig. 1, and the region corresponding to the case of a purely imaginary phase factor $\varphi(x)$, i.e., to conditions which forbid the propagation of waves, is cross-hatched.

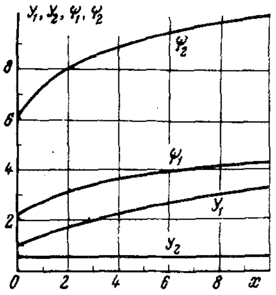

Fig. 9

Just as in the theory of nonlinear oscillations, it is not difficult to construct the direction field, i.e., the system of curves $F = F(A)$, by constructing, through arbitrarily chosen points situated in the region which allows wave propagation, a system of straight segments whose angle of inclination relative to the OA axis is determined at every point of the plane AOF by (4.2). The form of the boundary trajectory should be taken into account in order to render the construction easier.

By way of example, we shall construct in the plane AOF of Fig. 1, one of the curves of the family $F = F(A)$ for which we have $F = 0.5$, 0.25 and ∞ when $A = 1.2$ and 0, and map it onto the plane xOA. In order to do this, according to Fig. 2 we transfer this curve from the AOF system of coordinates to the x'OA system of coordinates and employ the relations $A' = F$ which is obvious by definition. Consequently, the function $F^{-1} = x$ will, for the curve selected, be equal to 2.0; 4.0 and 0, respectively, and the curve of Fig. 1 will map into the straight line in Fig. 2 passing through the origin of coordinates.

In order to map the straight line of Fig. 2 drawn on the intermediate plane x' OA into a curve on the plane xOA, dictated by the conditions of the wave propagation problem, it must be integrated with respect to the variable A which has been temporarily introduced, and so, leaving the vertical axis unchanged, we pass from x' to x on the horizontal axis in accordance with the rules of graphical integration.

Since the integration is carried out for arbitrary initial conditions, it is possible to obtain a series of other curves in addition to the curve $x = x(A)$ which has been drawn, and which is at the same time the graph of the function $A = A(x)$. All these curves will differ from each other only in the magnitude of their displacement along the Ox.

The original function $k = k(x)$ is constructed from the curve $A = A(x)$, which has been found, and the graph $k = k(A)$, which is given, in accordance with Fig. 3, where, in addition to the integral curve 1 already considered, curves 2,3 and 4 are also drawn, for which the constant coefficients characterizing the function $k = (ax + b)^{-1}$ are equal to $a = 1, 1, -1, -1$ and $b = 0, 1, 5, 10$, respectively.

The general factor $Y_1e^{i\psi_1}$ of the solution (1.2) of Eq. (0.1), corresponding to the chosen function $k = k(A)$ is determined in the following manner. On the plane AOφ of Fig. 4 the curve $\varphi = \varphi(A)$ is drawn, having been found by graphical integration of the curve $\varphi'(A)$ constructed according to formula (2.9), which it is convenient to transcribe in the following manner for this purpose:

$$\varphi' = A^{-1}[(AF)^{-2} - 1]^{1/2}. \tag{4.4}$$

In carrying out the construction it should be kept in mind that the initial conditions in the integration are taken into account by the choice of constants C_1 and C_2 in (1.1). Further the parameters M(A) and N(A) are calculated from (2.11). For the case of (4.1), in particular, we obtain

$$M(A) = \{[1-(AF)^2]^{1/2} - 1\}(A^2F)^{-1}, \quad (4.5)$$

$$N(A) = -A^{-1}. \quad (4.6)$$

Finally, graphs of the functions $Y_1 = Y_1(A)$ and $\Psi_1 = \Psi_1(A)$ are constructed in Fig. 5 using formulas (2.12).

In order to find the general solution $\Psi(x)$ of the wave equation (0.1) we must also find the functions $Y_2 = Y_2(A)$, which is done by drawing graphs of the functions $R_1 = R_1(A)$ and $R_2 = R_2(A)$ from (2.14), and subsequently integrating them with respect to the variable A. The respective operations, and also the curves for the functions $J_1(A)$ and $J_2(A)$, are presented in Figs. 6 and 7. The required functions $Y_2 = Y_2(A)$ and $\Psi_2 = \Psi_2(A)$, calculated from (2.13), are drawn in Fig. 8.

In carrying out the final, third stage of the solution, i.e., in passing from A to x, it is necessary to be specific and choose one particular curve from the family of curves $k = k(x)$ (Fig. 3), constructed with the help of the curve Fig. 1, for example, curve 2, and also the graph of the function $A = A(x)$ corresponding to the curve chosen. The latter is drawn in Fig. 2 as a still unaveraged broken line, and is, moreover, represented in the first square of Fig. 3. Here the transformation of the variable reduces (since in accordance with the graphs of Fig. 3, a completely determined value of the coordinate x corresponds to each value of A(x)) to replacing the argument A of the functions $Y_1 = Y_1(A)$, $\Psi_1 = \Psi_1(A)$, $Y_2 = Y_2(A)$ and $\Psi_2 = \Psi_2(A)$ by the argument x.

The functions $Y_1 = Y_1(x)$, $\Psi_1 = \Psi_2(x)$, $Y_2 = Y_2(x)$ and $\Psi_2 = \Psi_2(x)$ constructed and drawn in Fig. 9 completely determine the solution of the problem of wave propagation in an inhomogeneous medium, for which the law (4.1) of variation of the parameter along the axis of the coordinate x is described by curve 2 of the fourth square of Fig. 3.

5. Analysis of the asymptotic method and comparison with the exact solution of the propagation problem. We shall carry out a comparison of the proposed asymptotic and exact methods using the analytic variant of the asymptotic method, i.e., without recourse to graphical construction. Considering the example which has just been solved for $k(A) = A^{-2}$, we shall, for this purpose, first of all determine the function $A = A(x)$ starting from (4.2). Integrating the equation $F' = -FA^{-1}$, we obtain

$$F = {}^1/_2 aA^{-1}. \quad (5.1)$$

Further, taking into account that $F = A'$ in Eq. (5.1) and integrating it, we obtain

$$A = (ax+b)^{1/2}. \quad (5.2)$$

Here $1/2a$ and b are consants of integration. Thus the given function $k = k(A) = A^{-2}$ corresponds to the following dependence of the parameter of the inhomogeneous medium on the coordinate:

$$k(x) = k[A(x)] = (ax+b)^{-1}. \quad (5.3)$$

Comparison of the relation obtained (5.3) with the graph Fig. 3 attests to the fact that the constant of integration b characterizes the magnitude of the displacement of the curve $k = k(x)$ along the horizontal axis. As has already been noted, this displacement is determined only by the inital conditions assumed in the graphical integration of the function $A = A(x')$, plotted in Fig. 2 in the form of a straight line. Whence it follows that when integrating graphically with respect to the variable A the lower limit must be set equal to $b^{1/2}$, then the choice of one or other of the family of curves $F = F(A)$ drawn in Fig. 1 and characterized by the given function $k = k(A)$ serves as the constant of integration a. On passing to the coordinates A, x' this now corresponds to the function $A = A(x')$.

In particular, the choice made in Fig. 1 of the curve passing through the point with coordinates $F = 0.5$, $A = 1$, (in solving the problem by the grapho-analytic variant) is equivalent to choosing $a = 1$ in (5.3). When determining the phase of the direct partial wave by integrating (4.4) with respect to the variable A and subsequently taking (5.1) into account we write

$$\varphi(A) = 2s\ln A + \varphi_0 \qquad (s = {}^1/_2[4a^{-2} - 1]^{1/2}).$$

In accordance with (4.5) the parameter M(A) is determined by the relation

$$M(A) = 2(s - a^{-1})A^{-1}$$

and N(A) by the relation (4.6). Thus the particular solution of Eq. (0.1) has the form

$$[\alpha(x) + \beta(x)] = -A\left[\frac{4}{a}\left(\frac{2}{a} - s\right)\right]^{1/2}\exp\left\{i\left[2s\ln A + \varphi_0 + \operatorname{arctg} 2\left(\frac{1}{a} - s\right)\right]\right\}.$$

Taking (5.2) into account, we rewrite this expression:

$$[\alpha(x) + \beta(x)] = D(ax+b)^{1/2}\exp[is\ln(ax+b)]$$

$$D = -\left[\frac{4}{a}\left(\frac{2}{a} - s\right)\right]^{1/2}\exp\left\{i\left[\varphi_0 + \operatorname{arc\,tg} 2\left(\frac{1}{a} - s\right)\right]\right\}.$$

Passing to the general solution (1.1), we obtain

$$\psi(x) = (ax+b)^{1/2}\{C_1\exp[is\ln(ax+b)] + C_2\exp[-is\ln(ax+b)]\} \quad (5.4)$$

where C_1 and C_2 are constants of integration.

The validity of the solution which has just been found may easily be checked by direct substitution of (5.4) and (5.3) in the original equation (0.1).

The example which has been considered shows that it is possible to apply both the grapho-analytic and the analytic variant of the asymptotic method of solving the wave propagation problem in an inhomogeneous medium. In the second case the appropriate transformations must be carried out without recourse to graphical constructions, and in those cases when a fairly complex function $k = k(A)$ is given numerical integration must be applied.

Since the system of calculation for the proposed method, as distinct from familiar approximate methods in the theory of inhomogeneous media, does not require any simplifying assumptions whatever, its accuracy is determined only by the accuracy of the calculations carried out in performing the elementary operations enumerated above (integration and transformation of the variable).

Solutions of the wave propagation problem in an inhomogeneous medium have been obtained by the asymptotic method just considered for more complicated cases including a periodic dependence of the parameter on the coordinate, but are not given here due to lack of space.

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